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Darboux transformation and soliton-like solutions for the Gerdjikov–Ivanov equation

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Abstract. An explicit N-fold Darboux transformation with multiparameters for a kind of coupled derivative nonlinear Schrödinger equation is constructed with the help of a gauge transformation of a spectral problem. As a reduction, a Darboux transformation of the Gerdjikov–Ivanov equation is obtained. Furthermore, the explicit soliton-like solutions of the Gerdjikov–Ivanov equation are given by applying its Darboux transformation.

1. Introduction

The nonlinear Schrödinger (NLS) equation is one of the most generic soliton equations, and arises from a wide variety of fields, such as quantum field theory, weakly nonlinear dispersive water waves and nonlinear optics [1–3]. To study the effect of higher-order perturbations, various modifications and generalizations of the NLS equations have been proposed and studied [3–10]. Among them, there are three celebrated equations with derivative-type nonlinearities, which are called the derivative nonlinear Schrödinger (DNLS) equations. One is the Kaup–Newell equation [4]

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}(|q|^2 q)_x = 0$$

which is usually called DNLSI. The second type is the Chen–Lee–Liu equation [5, 6]

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}|q|^2 q_x = 0$$

which is called DNLSII. The last one takes the form

$$\mathrm{i}q_t + q_{xx} - \mathrm{i}q^2 q_x^* + \tfrac{1}{2}q^3 q^{*2} = 0 \tag{1.1}$$

which is called the Gerjikov–Ivanov (GI) equation or DNLSIII [7, 8]. In equation (1.1), q^* denotes the complex conjugation of q. It is known that these three equations may be transformed into each other by a gauge transformation, and the method of gauge transformation can also be applied to some generalized cases [8, 10–12]. In recent years, the spectral problem, Hamiltonian structure, Painlevé property, exact solutions and other properties associated with the Kaup–Newell equation have been investigated in detail [3, 4, 13–15]. Therefore, the corresponding results for the GI equation (1.1) may be obtained in principle by some gauge transformation [6, 8]. However, to obtain their explicit forms, one must solve an integrable equation in practice (for example, see [6]). This integration will become very complicated with increasing iteration times, especially in the multi-soliton case.

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In this paper, we study the GI equation (1.1) with the help of a spectral problem and the Darboux matrix method. The Darboux transformation (DT) has proved to be one of most fruitful algorithmic procedures to obtain explicit solutions of partial differential equations [16–21]. Its advantage is that the new solutions can be obtained successively by using an algebraic algorithm. This paper is organized as follows. In the next section, we shall establish an explicit N-fold DT with multi-parameter for a generalized Kaup–Newell spectral problem and the associated coupled DNLS equations (see (2.1) and (2.2) below) by using a systematic procedure. In section 3, an explicit Darboux transformation of the GI equation (1.1) is obtained through a reduction technique. Furthermore, the one- and two-soliton solutions of GI equation (1.1) are given explicitly by applying its DT.

The DT presented in this paper has some merits. It can be interpreted as a nonlinear superposition of the initial solution and the N-soliton solution of the GI equation (1.1), and it contains all pure N-soliton solutions. Moreover, from this DT the solutions of the GI equation are reduced to solving a linear algebraic system. It is very easy to produce their multi-soliton solutions by symbolic computation on a computer.

2. Darboux transformation

In this section, we shall construct a DT for the coupled DNLS equations

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}q^2 r_x + \tfrac{1}{2}q^3 r^2 = 0 \tag{2.1}$$

$$\mathrm{i}r_t - r_{xx} + \mathrm{i}r^2 q_x - \tfrac{1}{2}q^2 r^3 = 0. \tag{2.2}$$

Equations (2.1) and (2.2) are exactly reduced to the GI equation (1.1) for the choice $r = -q^*$.

The Lax pairs corresponding to coupled DNLS equations (2.1) and (2.2) can be given by the generalized Kaup–Newell spectral problem

$$\begin{gathered}\psi_x = U\psi \qquad \psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\\ U = \begin{pmatrix}-\mathrm{i}\lambda^2 - \frac{1}{2}\mathrm{i}qr & \lambda q\\ \lambda r & \mathrm{i}\lambda^2 + \frac{1}{2}\mathrm{i}qr\end{pmatrix}\end{gathered} \tag{2.3}$$

and the auxiliary problem [15]

$$\psi_t = V\psi \qquad V = \begin{pmatrix}a & b\\ c & -a\end{pmatrix} \tag{2.4}$$

with

$$a = -2\mathrm{i}\lambda^4 - \mathrm{i}qr\lambda^2 + \tfrac{1}{2}(rq_x - qr_x) + \tfrac{1}{4}\mathrm{i}q^2r^2$$

$$b = 2q\lambda^3 + \mathrm{i}q_x\lambda \qquad c = 2r\lambda^3 - \mathrm{i}r_x\lambda.$$

where q and r are two potentials, and λ is a spectral parameter.

It is easy to see that the Lax pairs (2.3) and (2.4) are transformed to

$$\tilde{\psi}_x = \tilde{U}\tilde{\psi} \qquad \tilde{U} = (T_x + TU)T^{-1} \tag{2.5}$$

$$\tilde{\psi}_t = \tilde{V}\tilde{\psi} \qquad \tilde{V} = (T_t + TV)T^{-1} \tag{2.6}$$

under a gauge transformation

$$\tilde{\psi} = T\psi. \tag{2.7}$$

By cross differentiating (2.5) and (2.6), we obtain

$$\tilde{U}_t - \tilde{V}_x + [\tilde{U}, \tilde{V}] = T(U_t - V_x + [U, V])T^{-1}$$

which implies that in order to make equations (2.1) and (2.2) invariant under the transformation (2.7), it is crucial to find a matrix T such that $\tilde{U}$, $\tilde{V}$ have the same forms as U, V. At the same time the old potentials q and r in U, V are mapped into new potentials $\tilde{q}$ and $\tilde{r}$ in $\tilde{U}$, $\tilde{V}$.

Let the Darboux matrix T in (2.7) be in the form

$$T = T(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.8}$$

where

$$A = \lambda^N + \sum_{k=0}^{N-1} A_k \lambda^k \qquad B = \sum_{k=0}^{N-1} B_k \lambda^k$$

$$C = \sum_{k=0}^{N-1} C_k \lambda^k \qquad D = \lambda^N + \sum_{k=0}^{N-1} D_k \lambda^k.$$

A_k, B_k, C_k and D_k are given by a linear algebraic system

$$\sum_{k=0}^{N-1} (A_k + B_k \alpha_j)\lambda_j^k = -\lambda_j^N \qquad \sum_{k=0}^{N-1} (C_k + D_k \alpha_j)\lambda_j^k = -\alpha_j \lambda_j^N \tag{2.9}$$

with

$$\alpha_j = \frac{\phi_2(\lambda_j) - \gamma_j \psi_2(\lambda_j)}{\phi_1(\lambda_j) - \gamma_j \psi_1(\lambda_j)} \qquad 1 \leqslant j \leqslant 2N \tag{2.10}$$

where $\phi = (\phi_1, \phi_2)^T$, $\psi = (\psi_1, \psi_2)^T$ are two basic solutions of the spectral problem (2.3) and (2.4), λ_j and γ_j ($\lambda_k \neq \lambda_j$, $\gamma_k \neq \gamma_j$, as $k \neq j$) are some parameters suitably chosen such that the determinant of coefficients for (2.9) are non-zero. Therefore, A_k, B_k, C_k and D_k ($0 \leqslant k \leqslant N-1$) are uniquely determined by (2.9).

Equation (2.8) shows that $\det T(\lambda)$ is a $(2N)$th-order polynomial of λ, and

$$\det T(\lambda_j) = A(\lambda_j)D(\lambda_j) - B(\lambda_j)C(\lambda_j).$$

On the other hand, from (2.9) we have

$$A(\lambda_j) = -\alpha_j B(\lambda_j) \qquad C(\lambda_j) = -\alpha_j D(\lambda_j). \tag{2.11}$$

Therefore, it holds that

$$\det T(\lambda_j) = 0$$

which implies that λ_j ($1 \leqslant j \leqslant 2N$) are $2N$ roots of $\det T(\lambda)$, that is,

$$\det T(\lambda) = \prod_{j=1}^{2N} (\lambda - \lambda_j). \tag{2.12}$$

By using the above facts, we can prove the following proposition:

Proposition 1. *The matrix $\tilde{U}$ determined by (2.5) has the same form as U, that is,*

$$\tilde{U} = \begin{pmatrix} -\mathrm{i}\lambda^2 - \frac{1}{2}\mathrm{i}\tilde{q}\tilde{r} & \lambda\tilde{q} \\ \lambda\tilde{r} & \mathrm{i}\lambda^2 + \frac{1}{2}\mathrm{i}\tilde{q}\tilde{r} \end{pmatrix}$$

where the transformations between q, r and $\tilde{q}$, $\tilde{r}$ are given by

$$\tilde{q} = q + 2\mathrm{i}B_{N-1} \qquad \tilde{r} = r - 2\mathrm{i}C_{N-1}. \tag{2.13}$$

The transformation (2.7) and (2.13): $(\psi, q, r) \to (\tilde{\psi}, \tilde{q}, \tilde{r})$ is called a DT of the spectral problem (2.3).

Proof. Let $T^{-1} = T^*/\det T$ and

$$(T_x + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix}. \tag{2.14}$$

It is easy to see that $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are $(2N+2)$th-order polynomials in λ, $f_{12}(\lambda)$ and $f_{21}(\lambda)$ are $(2N+1)$th-order polynomials in λ. From (2.3) and (2.10), we find that

$$\alpha_{jx} = \lambda_j r - \lambda_j q\alpha_j^2 + 2\mathrm{i}\left(\lambda_j^2 + \tfrac{1}{2}qr\right)\alpha_j. \tag{2.15}$$

By using (2.11) and (2.15), we can verify that all λ_j $(1 \leqslant j \leqslant 2N)$ are roots of $f_{kj}(\lambda)$ $(k, j = 1, 2)$. Again noting (2.12), then we conclude that

$$\det T \,|\, f_{kj}(\lambda) \qquad k, j = 1, 2$$

which, together with (2.14) gives

$$(T_x + TU)T^* = (\det T)P(\lambda) \tag{2.16}$$

with

$$P(\lambda) = \begin{pmatrix} p_{11}^{(2)}\lambda^2 + p_{11}^{(1)}\lambda + p_{11}^{(0)} & p_{12}^{(1)}\lambda + p_{12}^{(0)} \\ p_{21}^{(1)}\lambda + p_{21}^{(0)} & p_{22}^{(2)}\lambda^2 + p_{22}^{(1)}\lambda + p_{22}^{(0)} \end{pmatrix}$$

where $p_{kj}^{(l)}$ $(k, j = 1, 2, l = 0, 1, 2)$ are undetermined functions independent of λ. Now equation (2.16) can be written in the form

$$T_x + TU = P(\lambda)T. \tag{2.17}$$

By comparing the coefficients of λ^{N+2}, λ^{N+1} and λ^N in (2.17), we obtain

$$p_{11}^{(1)} = p_{22}^{(1)} = p_{12}^{(0)} = p_{21}^{(0)} = 0 \qquad p_{11}^{(2)} = -p_{22}^{(2)} = -\mathrm{i}$$

$$p_{12}^{(1)} = q + 2\mathrm{i}B_{N-1} = \tilde{q} \qquad p_{21}^{(1)} = r - 2\mathrm{i}C_{N-1} = \tilde{r}$$

$$p_{11}^{(0)} = -\tfrac{1}{2}\mathrm{i}qr + rB_{N-1} - p_{12}^{(1)}C_{N-1} = -\tfrac{1}{2}\mathrm{i}\tilde{q}\tilde{r}$$

$$p_{22}^{(0)} = \tfrac{1}{2}\mathrm{i}qr + qC_{N-1} - p_{12}^{(1)}B_{N-1} = \tfrac{1}{2}\mathrm{i}\tilde{q}\tilde{r}.$$

From (2.5) and (2.17), we see that $\tilde{U} = P(\lambda)$. The proof is completed. □

Let the ϕ and ψ also satisfy equation (2.4), we try to prove that $\tilde{V}$ in (2.6) has the same form as V under the transformation (2.7) and (2.13).

Proposition 2. *Under DT (2.7) and (2.13), the matrix $\tilde{V}$ in (2.6) has the same form as V, that is,*

$$\tilde{V} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & -\tilde{a} \end{pmatrix} \tag{2.18}$$

in which

$$\tilde{a} = -2\mathrm{i}\lambda^4 - \mathrm{i}\tilde{q}\tilde{r}\lambda^2 + \tfrac{1}{2}(\tilde{r}\tilde{q}_x - \tilde{q}\tilde{r}_x) + \tfrac{1}{4}\mathrm{i}\tilde{q}^2\tilde{r}^2$$

$$\tilde{b} = 2\tilde{q}\lambda^3 + \mathrm{i}\tilde{q}_x\lambda \qquad \tilde{c} = 2\tilde{r}\lambda^3 - \mathrm{i}\tilde{r}_x\lambda.$$

The old potentials q and r are mapped into new ones $\tilde{q}$ and $\tilde{r}$ according to the same DT (2.7) and (2.13).

Proof. In a way similar to theorem 1, we denote $T^{-1} = T^*/\det T$ and

$$(T_t + TV)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}. \tag{2.19}$$

Direct calculation shows that $g_{11}(\lambda)$, $g_{22}(\lambda)$ or $g_{12}(\lambda)$, $g_{21}(\lambda)$ are $(2N+4)$th or $(2N+3)$th polynomials in λ, respectively. With the help of (2.4), (2.10) and (2.11), we find that

$$\alpha_{jt} = c(\lambda_j) - 2a(\lambda_j)\alpha_j - b(\lambda_j)\alpha_j^2 \tag{2.20}$$

$$A_t(\lambda_j) = -B_t(\lambda_j)\alpha_j - B(\lambda_j)\alpha_{jt} \tag{2.21}$$

$$C_t(\lambda_j) = -D_t(\lambda_j)\alpha_j - D(\lambda_j)\alpha_{jt}. \tag{2.22}$$

We can verify by (2.11) and (2.20)–(2.22) that λ_j $(1 \leqslant \lambda_j \leqslant 2N)$ are also roots of $g_{kj}(\lambda)$ $k, j = 1, 2$. Therefore, we have

$$\det T(\lambda) | g_{kj}(\lambda) \qquad k, j = 1, 2.$$

and thus

$$(T_t + TV)T^* = (\det T)Q(\lambda)$$

with

$$Q(\lambda) = \begin{pmatrix} q_{11}^{(4)}\lambda^4 + q_{11}^{(3)}\lambda^3 + q_{11}^{(2)}\lambda^2 + q_{11}^{(1)}\lambda + q_{11}^{(0)} & q_{12}^{(3)}\lambda^3 + q_{12}^{(2)}\lambda^2 + q_{12}^{(1)}\lambda + q_{12}^{(0)} \\ q_{21}^{(3)}\lambda^3 + q_{21}^{(2)}\lambda^2 + q_{21}^{(1)}\lambda + q_{21}^{(0)} & q_{22}^{(4)}\lambda^4 + q_{22}^{(3)}\lambda^3 + q_{22}^{(2)}\lambda^2 + q_{22}^{(1)}\lambda + q_{22}^{(0)} \end{pmatrix}$$

that is

$$T_t + TV = Q(\lambda)T \tag{2.23}$$

Comparing the coefficients of λ^{N+4}, λ^{N+3}, λ^{N+2}, λ^{N+1} and λ^N in (2.23) leads to

$$q_{11}^{(3)} = q_{11}^{(1)} = q_{22}^{(3)} = q_{22}^{(1)} = q_{12}^{(2)} = q_{12}^{(0)} = q_{21}^{(2)} = q_{21}^{(0)} = 0.$$

$$q_{11}^{(4)} = -q_{22}^{(4)} = -2\mathrm{i} \qquad q_{12}^{(3)} = 2\tilde{q} \qquad q_{21}^{(3)} = 2\tilde{r}$$

$$q_{11}^{(2)} = -\mathrm{i}qr + 2rB_{N_1} - q_{12}^{(3)}C_{N-1} = -\mathrm{i}\tilde{q}\tilde{r}$$

$$q_{22}^{(2)} = \mathrm{i}qr + 2qC_{N_1} - q_{21}^{(3)}B_{N-1} = \mathrm{i}\tilde{q}\tilde{r}$$

$$q_{12}^{(1)} = \mathrm{i}q_x + 4\mathrm{i}B_{N-3} + 2qA_{N-2} - 2\tilde{q}D_{N-2} + \mathrm{i}qrB_{N-1} + \mathrm{i}\tilde{q}\tilde{r}B_{N-1} \tag{2.24}$$

$$q_{21}^{(1)} = -\mathrm{i}r_x - 4\mathrm{i}C_{N-3} + 2rD_{N-2} - 2\tilde{r}A_{N-2} - \mathrm{i}qrC_{N-1} - \mathrm{i}\tilde{q}\tilde{r}C_{N-1} \tag{2.25}$$

$$q_{11}^{(0)} = \frac{1}{2}(rq_x - qr_x) + \frac{1}{4}\mathrm{i}q^2r^2 - \mathrm{i}r_x B_{N-1} - \mathrm{i}\tilde{q}_x C_{N-1} - \mathrm{i}r(2\mathrm{i}B_{N-3} + qA_{N-2} - \tilde{q}D_{n-2}) \\ -\mathrm{i}\tilde{q}(-2\mathrm{i}C_{N-3} - \tilde{r}A_{N-2} + rD_{N-2}) \tag{2.26}$$

$$q_{22}^{(0)} = -\frac{1}{2}(rq_x - qr_x) - \frac{1}{4}\mathrm{i}q^2r^2 + \mathrm{i}q_x C_{N-1} + \tilde{r}_x B_{N-1} + \mathrm{i}\tilde{r}(2\mathrm{i}B_{N-3} + qA_{N-2} - \tilde{q}D_{N-2}) \\ +\mathrm{i}q(-2\mathrm{i}C_{N-3} - \tilde{r}A_{N-2} + rD_{N-2}). \tag{2.27}$$

On the other hand, equating the coefficients of λ^{N-1} in (2.17) gives

$$2\mathrm{i}B_{N-3} + qA_{N-2} - \tilde{q}D_{N-2} = -B_{N-1x} - \frac{1}{2}\mathrm{i}qrB_{N-1} - \frac{1}{2}\mathrm{i}\tilde{q}\tilde{r}B_{N-1} \tag{2.28}$$

$$-2\mathrm{i}C_{N-3} + rD_{N-2} - \tilde{r}A_{N-2} = -C_{N-1x} + \frac{1}{2}\mathrm{i}qrC_{N-1} + \mathrm{i}\tilde{q}\tilde{r}C_{N-1}. \tag{2.29}$$

On substituting (2.28) and (2.29) into (2.24)–(2.27), direct calculations show that

$$q_{12}^{(1)} = \mathrm{i}\tilde{q}_x \qquad q_{21}^{(1)} = -\mathrm{i}\tilde{r}_x \qquad q_{11}^{(0)} = -q_{22}^{(0)} = \frac{1}{2}(\tilde{r}\tilde{q}_x - \tilde{q}\tilde{r}_x) + \frac{1}{4}\mathrm{i}\tilde{q}^2\tilde{r}^2.$$

Then (2.18) is obtained from (2.6) and (2.23). The proof is completed. □

Proposition 1 and 2 show that the transformation (2.7) and (2.13) changes the Lax pairs (2.3) and (2.4) into another set of Lax pairs (2.5) and (2.6) of the same type. Therefore, both of the Lax pairs lead to the same coupled DNLS equations (2.1) and (2.2). We call the transformation $(\psi, q, r) \rightarrow (\tilde{\psi}, \tilde{q}, \tilde{r})$ a DT of coupled DNLS equations (2.1) and (2.2). In summary, we arrive at

Theorem 1. *The solutions (q, r) of coupled DNLS equations (2.1) and (2.2) are mapped into their new solutions $(\tilde{q}, \tilde{r})$ under the DT (2.7) and (2.13).*

3. Reduction of Darboux transformation and application

In this section, we discuss the DT of the GI equation (1.1) and give its explicit solutions. For this purpose we let $r = -q^*$, and choose two solutions of the Lax pairs (2.3) and (2.4)

$$\phi(\lambda) = (\phi_1(\lambda), \phi_2(\lambda))^T \qquad \psi(\lambda) = (-\phi_2^*(\lambda^*), \phi_1^*(\lambda^*))^T$$

and parameters

$$\lambda_{2j} = \lambda_{2j-1}^* \qquad \gamma_{2j} = -\gamma_{2j-1}^{*-1} \qquad 1 \leqslant j \leqslant N.$$

Then it is easy to show that $\alpha_{2j}^{-1} = -\alpha_{2j-1}^*$, $D_k^* = A_k$, $C_k^* = -B_k$ $(0 \leqslant k \leqslant N-1)$. In this way, the solutions corresponding to (2.9) and (2.10) are reduced to

$$\sum_{k=0}^{N-1}(A_k + \alpha_{2j-1}B_k)\lambda_{2j-1}^k = -\lambda_{2j-1}^N \tag{3.1}$$

$$\sum_{k=0}^{N-1}(\alpha_{2j-1}^*A_k - B_k)\lambda_{2j-1}^{*k} = -\alpha_{2j-1}^*\lambda_{2j-1}^{*N} \qquad 1 \leqslant j \leqslant N. \tag{3.2}$$

and

$$\alpha_{2j-1} = \frac{\phi_2(\lambda_{2j-1}) - \gamma_{2j-1}\psi_2(\lambda_{2j-1})}{\phi_1(\lambda_{2j-1}) - \gamma_{2j-1}\psi_1(\lambda_{2j-1})}. \tag{3.3}$$

Now we have the following theorem:

Theorem 2. *Suppose that* α_{2j-1} $(1 \leqslant j \leqslant N)$ *is defined by (3.3), and* A_k, B_k *are given by the linear algebraic system (3.1) and (3.2). Then the solution* q *of the GI equation (1.1) is mapped into its a new solution* $\tilde{q}$ *under the DT*

$$\tilde{q} = q + 2\mathrm{i}B_{N-1}. \tag{3.4}$$

In the following, we shall apply the DT to construct explicit solutions of the GI equation (1.1). As usual we make DT starting from a special solution of equation (1.1). Substituting $q = 0$ into the Lax pairs (2.3) and (2.4), we find that two basic solutions can be chosen as

$$\phi(\lambda) = \begin{pmatrix} \exp(-\mathrm{i}\lambda^2 x - 2\mathrm{i}\lambda^4 t) \\ 0 \end{pmatrix} \qquad \psi(\lambda) = \begin{pmatrix} 0 \\ \exp(\mathrm{i}\lambda^2 x + 2\mathrm{i}\lambda^4 t) \end{pmatrix}.$$

According to (3.3), we have

$$\alpha_{2j-1} = -\exp(2\mathrm{i}\lambda_{2j-1}^2 x + 4\mathrm{i}\lambda_{2j-1}^4 t + \delta_j + \mathrm{i}\mu_j) \qquad 1 \leqslant j \leqslant N \tag{3.5}$$

where $\gamma_{2j-1} = \exp(\delta_j + \mathrm{i}\mu_j)$. For simplicity, we shall discuss the two special cases $N = 1$ and 2.

(I) For $N = 1$, let $\lambda_1 = \xi_1 + \mathrm{i}\eta_1$ $(\xi_1 \neq \eta_1)$. Then solving the linear algebraic system (3.1) and (3.2) leads to

$$B_0 = \mathrm{i}\eta_1 \exp(\mathrm{i}Y_1)\,\mathrm{sech}(X_1)$$

where

$$\begin{aligned} X_1 &= -4\xi_1\eta_1 x - 16\xi_1\eta_1(\xi_1^2 - \eta_1^2)t + \delta_1 \\ Y_1 &= 2(\xi_1^2 - \eta_1^2)x + 4(\xi_1^4 + \eta_1^4 - 6\xi_1^2\eta_1^2)t + \mu_1. \end{aligned} \tag{3.6}$$

In this way, a one-soliton solution of the GI equation (1.1) is obtained with the help of the DT (3.4)

$$\tilde{q} = 2\mathrm{i}B_0. \tag{3.7}$$

The plots of this solution are given in figure 1.

(II) For $N = 2$, let $\lambda_1 = \xi_1 + \mathrm{i}\eta_1$, $\lambda_3 = \xi_2 + \mathrm{i}\eta_2$ $(\xi_1 \neq \xi_2)$. Then we obtain from (3.5)

$$\alpha_1 = -\exp(X_1 + \mathrm{i}Y_1) \qquad \alpha_3 = -\exp(X_2 + \mathrm{i}Y_2)$$

in which X_1, Y_1 are defined by (3.6), and X_2, Y_2 are given by

$$\begin{aligned} X_2 &= -4\xi_2\eta_2 x - 16\xi_2\eta_2(\xi_2^2 - \eta_2^2)t + \delta_2 \\ Y_2 &= 2(\xi_2^2 - \eta_2^2)x + 4(\xi_2^4 + \eta_2^4 - 6\xi_2^2\eta_2^2)t + \mu_2. \end{aligned}$$

Solving the linear algebraic system (3.1) and (3.2) yields

$$B_1 = \frac{\Delta_{B_1}}{\Delta}$$

where Δ is the determinant of the coefficients for the linear algebraic system (3.1) and (3.2), and Δ_{B_1} is produced from Δ by replacing its fourth column with $(-\lambda_1^2, -\lambda_3^2, -\alpha_1^*\lambda_1^{*2}, -\alpha_3^*\lambda_3^{*2})^T$, that is

$$\Delta = \begin{vmatrix} 1 & \alpha_1 & \lambda_1 & \alpha_1\lambda_1 \\ 1 & \alpha_3 & \lambda_3 & \alpha_3\lambda_3 \\ \alpha_1^* & -1 & \alpha_1^*\lambda_1^* & -\lambda_1^* \\ \alpha_3^* & -1 & \alpha_3^*\lambda_3^* & -\lambda_3^* \end{vmatrix} \qquad \Delta_B = \begin{vmatrix} 1 & \alpha_1 & \lambda_1 & -\lambda_1^2 \\ 1 & \alpha_3 & \lambda_3 & -\lambda_3^2 \\ \alpha_1^* & -1 & \alpha_1^*\lambda_1^* & -\alpha_1^*\lambda_1^{*2} \\ \alpha_3^* & -1 & \alpha_3^*\lambda_3^* & -\alpha_3^*\lambda_3^{*2} \end{vmatrix}.$$

In this way, another solution of the GI equation (1.1) is obtained by using the DT (3.4)

$$\tilde{q} = 2\mathrm{i}B_1 \tag{3.8}$$

which is a two-soliton solutions. The plots are given in figure 2.

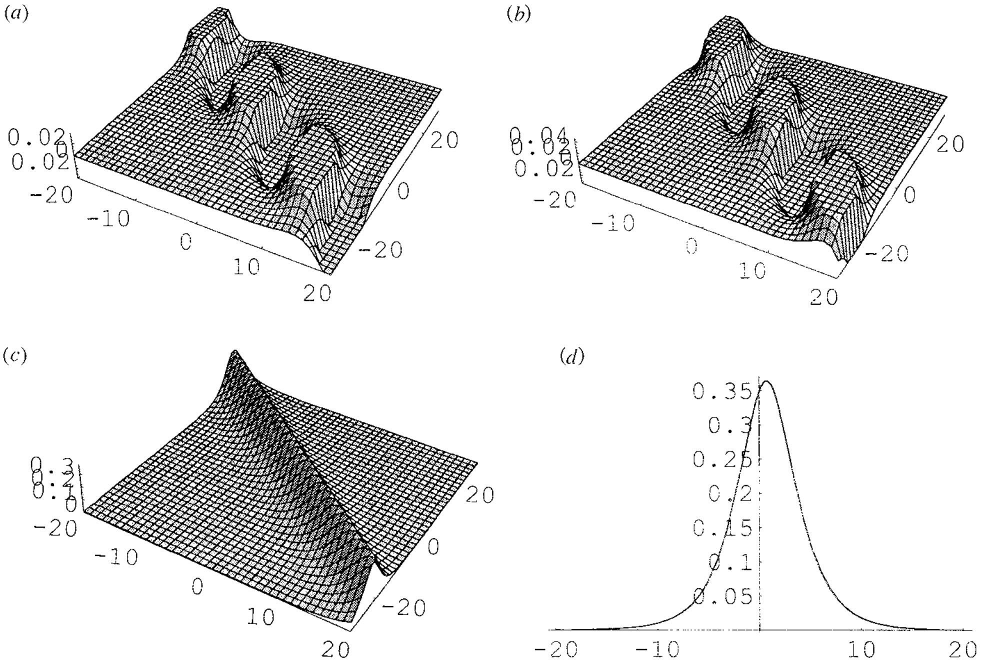

Figure 1. One-soliton solution (3.7) with $\xi_1 = 0.5$, $\eta_1 = 0.2$, $\delta_1 = 0.3$, $\mu_1 = 0.6$. (*a*) Real part of q, (*b*) imaginary part of q, (*c*) modulus of q, (*d*) q at $t = 0$.

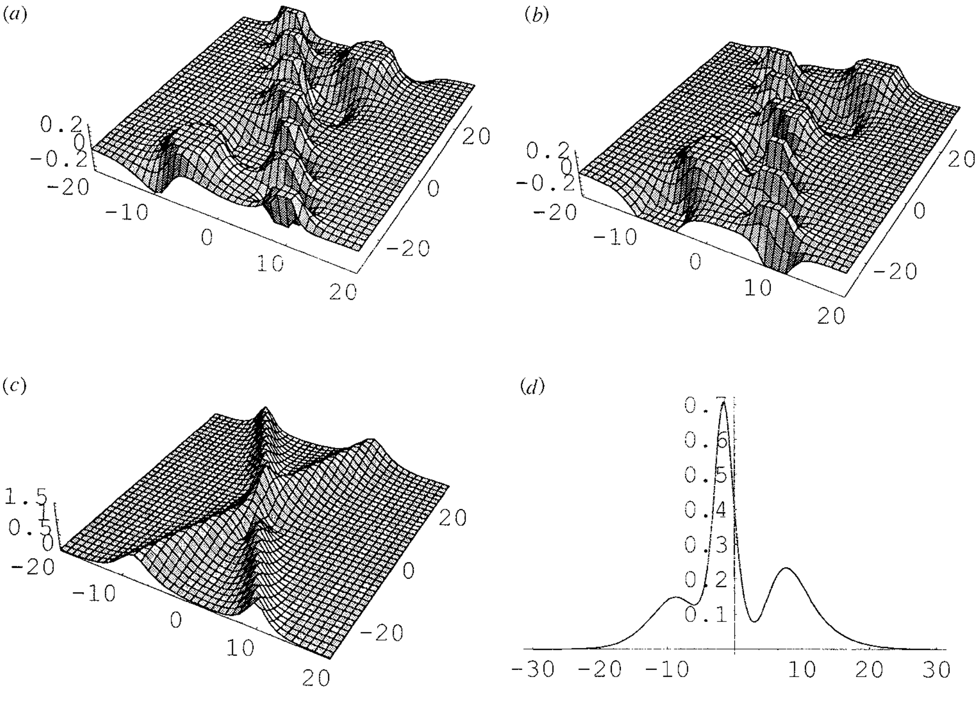

Figure 2. Two-soliton solution (3.8) with $\xi_1 = 0.5$, $\xi_2 = -0.3$, $\eta_1 = 0.2$, $\eta_2 = 0.1$, $\delta_1 = 0.3$, $\delta_2 = -0.1$, $\mu_1 = 0.6$, $\mu_2 = -0.2$. (*a*) Real part of q, (*b*) imaginary part of q, (*c*) modulus of q, (*d*) q at $t = 0$.

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